ANOSOV LIE ALGEBRAS AND ALGEBRAIC UNITS IN NUMBER FIELDS

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Abstract. We study nilmanifolds admitting Anosov automorphisms by applying elementary properties of algebraic units in number fields to the associated Anosov Lie algebras. We identify obstructions to the existence of Anosov Lie algebras. The case of 13-dimensional Anosov Lie algebras is worked out as an illustration of the technique. Also, we recapture the following known results: (i) Every 7-dimensional Anosov nilmanifold is toral, and (ii) every 8-dimensional Anosov Lie algebra with 3 or 5-dimensional derived algebra contains an abelian factor.

1. Introduction

A diffeomorphism f of a compact differentiable manifold M is said to be *Anosov* if there is a continuous invariant splitting of the tangent bundle $TM = E^+ \oplus E^-$ such that df expands E^+ and contracts E^- exponentially. Such diffeomorphisms are important in the study of hyperbolic dynamics. The only known examples of Anosov diffeomorphisms on compact manifolds are defined on nilmanifolds, or more generally, infranilmanifolds. We recall that a *nilmanifold* N/Γ is a compact quotient of a simply connected nilpotent Lie group N by a discrete subgroup $\Gamma \subset N$; an *infranilmanifold* is a manifold finitely covered by a nilmanifold. A. Manning (see [11]) and J. Franks (see [6]) proved that any Anosov diffeomorphism on a nilmanifold N/Γ is topologically conjugate to an *Anosov automorphism*, i.e. a diffeomorphism of N/Γ induced by a hyperbolic Lie group automorphism of N mapping Γ to itself. ("Hyperbolic" means that no eigenvalue of the differential of the automorphism is of absolute value 1.) In [14], S. Smale raised the problem of classifying the nilmanifolds admitting Anosov diffeomorphisms, which in view of Manning and Franks's results reduces to the classification of nilmanifolds with Anosov automorphisms.

At the level of Lie Algebras, this problem corresponds to the classification of *Anosov* Lie algebras, i.e., the Lie algebras of nilpotent Lie groups which have nilmanifold quotients admitting Anosov automorphisms. For previous work in this direction, see [14, 1, 2, 3, 4, 5, 8, 9, 10, 12, 13]. In this paper we consider a number-theoretic approach, which gives some information regarding the structure of Anosov Lie algebras. We work out the case of 13 dimensional Lie algebras as an illustration of this method, and give simpler proofs of some results obtained in [9, 10].

To state our main result, we recall a few definitions introduced in [9].

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Definition 1. Let $\mathfrak{n}$ be a Lie algebra. An *abelian factor* of $\mathfrak{n}$ is an abelian (Lie) ideal $\mathfrak{a}$ of $\mathfrak{n}$ such that $\mathfrak{n} = \mathfrak{m} \oplus \mathfrak{a}$ for some ideal $\mathfrak{m}$ of $\mathfrak{n}$.

Definition 2. Let $\mathfrak{n}$ be an r-step nilpotent Lie algebra, i.e., the lower central series $\{C^i(\mathfrak{n})\}$ (defined by $C^0(\mathfrak{n}) = \mathfrak{n}$ and $C^i(\mathfrak{n}) = [\mathfrak{n}, C^{i-1}(\mathfrak{n})]$ for $i \geq 1$) satisfies $C^{r-1}(\mathfrak{n}) \neq 0$ and $C^r(\mathfrak{n}) = 0$. Then the *type* of $\mathfrak{n}$ is the r-tuple of positive integers $(n_1, \cdots, n_r)$, where $n_i = \dim C^{i-1}(\mathfrak{n})/C^i(\mathfrak{n})$.

We characterize the complexifications of 13-dimensional Anosov Lie algebras up to isomorphism:

Theorem 1. *Every 13-dimensional real Anosov Lie algebra without an abelian factor is of type* $(9, 4)$*. For each such Lie algebra there exist complex numbers* a, b, c, d *such that its complexification has* $\mathbb{C}$*-linear basis*

$$X_1, X_2, X_3, Y_1, \cdots, Y_6, Z_1, Z_2, W_1, W_2,$$

where the only non-zero brackets of basis elements are

$$\begin{array}{ll} [X_1, Y_1] = Z_1, & [X_2, Y_2] = W_1 \\ [X_1, Y_4] = Z_2, & [X_2, Y_5] = W_2 \\ [X_3, Y_3] = aZ_1 + bW_1, & [X_3, Y_6] = cZ_2 + dW_2. \end{array} \tag{1}$$

An actual example of a 13-dimensional Anosov Lie algebra was given in [13]. It will be seen later that the hypothesis in Theorem 1 regarding absence of an abelian factor is natural. Our method gives new proofs of the following results from [9, 10]:

Theorem 2.
1. *Every 7-dimensional Anosov Lie algebra is abelian.*
2. *Every* 8*-dimensional Anosov Lie algebra of type* $(5, 3)$ *has an abelian factor.*
3. *There are no* 8*-dimensional Anosov Lie algebras of type* $(3, 3, 2)$*.*

In the last section, we give a necessary condition for a Lie algebra of type $(n, 2)$ to be Anosov for n odd.

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2. Some Background

Following [8], a rational Lie algebra $\mathfrak{n}$ is said to be *Anosov* if it admits a hyperbolic automorphism τ (i.e. all eigenvalues of τ have absolute value different from 1), and there is a basis of $\mathfrak{n}$ with respect to which the matrix of τ has integer entries. We say that a real Lie algebra is Anosov if it admits a rational form which is Anosov. The map τ will be called an Anosov automorphism of the Lie algebra $\mathfrak{n}$.

If a nilmanifold N/Γ admits an Anosov automorphism, then it can be seen that the rational Lie algebra determined by the lattice Γ is Anosov and hence the Lie algebra of N is Anosov. Therefore, in order to study the nilmanifolds admitting Anosov automorphisms, we can equivalently study Anosov Lie algebras.

We recall [9, Theorem 3.1]: Let $\mathfrak{n}$ be a rational Lie algebra and let $\mathfrak{n} = \mathfrak{m} \oplus \mathfrak{a}$ be a Lie direct sum, where $\mathfrak{a}$ is a maximal abelian factor of $\mathfrak{n}$. Then $\mathfrak{n}$ is Anosov iff $\mathfrak{m}$ is Anosov

and dim $\mathfrak{a} \geq 2$. In view of this, we are interested in studying Anosov Lie algebras without an abelian factor.

Let τ be an Anosov automorphism of a Lie algebra $\mathfrak{n}$. Then the eigenvalues of τ are algebraic integers. This follows since by definition there is a distinguished basis with respect to which the matrix of τ has integer entries, so that the characteristic polynomial of τ is monic with integer coefficients. Since τ^{-1} is also an Anosov automorphism, it follows that the reciprocals of the eigenvalues of τ are also algebraic integers. It follows that *an eigenvalue of an Anosov automorphism of a Lie algebra is an algebraic unit with absolute value different from 1.* This condition will allow us to prove the non-existence of Anosov Lie algebras (and therefore nilmanifolds admitting Anosov automorphisms) in some cases.

We summarize the present state of knowledge regarding Anosov Lie algebras. There are no Anosov Lie algebras of dimension less than 6 (see [15]). There exists an *indecomposable* (not a Lie direct sum of smaller dimensional Lie algebras) 2-step Anosov Lie algebra of dimension n, for every integer $n \geq 6$, $n \neq 7$ (see [3, 13].) For $n \geq 17$ and $n = 6, 8, 10, 11, 14, 15$, the examples of indecomposable 2-step Anosov Lie algebras are given in [3] which are associated to the graphs. For $n = 9, 12$, see [8]. For $n = 16$ one can modify the constructions of [3] (see [13]). These Lie algebras are associated with certain graphs as in [3] or obtained by modifying such examples. However the existence for $n = 13$ has been proved by using properties of the algebraic units (see [13]).

3. Algebraic Units not on the Unit Circle

Denote by $[E : F]$ the degree of a field extension $F \subseteq E$. For $\lambda \in E$ let $F(\lambda)$ denote the smallest subfield of E containing λ and F. The degree of an algebraic element $\lambda \in E$ over F is denoted by $\deg_F(\lambda)$. We recall that $\deg_F(\lambda) = [F(\lambda) : F]$. We say that $\lambda' \in E$ is a *conjugate* of $\lambda \in E$ over F if λ and λ' satisfy the same minimal polynomial over F, and in this case λ and λ' are *conjugates over* F. We also note that the conjugate elements over F are conjugate under the action of the appropriate Galois group.

Remark 1. We note that $\mathbb{Q}(\alpha)(\beta) = \mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\alpha\beta)(\alpha) = \mathbb{Q}(\alpha\beta)(\beta)$ for all non-zero $\alpha, \beta \in \mathbb{C}$.

We call $\lambda \in \mathbb{C}$ an *algebraic unit* if it satisfies a monic polynomial with integer coefficients and with constant term ± 1. Note that λ is an algebraic unit iff both λ and λ^{-1} are algebraic integers.

Throughout this section we will assume that α and β are algebraic units and we will derive some properties of algebraic units which will be used to prove our main results.

We have

$$\begin{aligned} [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}] &= [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha\beta)][\mathbb{Q}(\alpha\beta) : \mathbb{Q}] \\ &= [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\beta)][\mathbb{Q}(\beta) : \mathbb{Q}]. \end{aligned} \tag{2}$$

Lemma 1. *If $deg_{\mathbb{Q}}(\beta) = deg_{\mathbb{Q}}(\alpha\beta) = deg_{\mathbb{Q}(\alpha)}(\beta)$ then $\alpha^{deg_{\mathbb{Q}}(\beta)} = \pm 1$.*

Proof. We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ containing α and β. Let $\deg_{\mathbb{Q}}(\beta) = n$. Let $\{\beta = \beta_1, \cdots, \beta_n\}$ denote the set of all conjugates of β over $\mathbb{Q}$ in $\overline{\mathbb{Q}}$ and let $\{\gamma_1 = \alpha\beta, \cdots, \gamma_n\}$ denote the set of all conjugates of $\alpha\beta$ over $\mathbb{Q}$. Note that the set of all conjugates of $\alpha\beta$ over $\mathbb{Q}$ is same as the set of all conjugates of $\alpha\beta$ over $\mathbb{Q}(\alpha)$ since $\deg_{\mathbb{Q}}(\alpha\beta) = \deg_{\mathbb{Q}(\alpha)}(\beta)$; see Remark 1. As noted above, the conjugates of $\alpha\beta$ (or of β) in $\overline{\mathbb{Q}}$ over $\mathbb{Q}(\alpha)$ are the conjugates under the action of the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}(\alpha)$. Hence $\beta, \alpha^{-1}\gamma_2, \ldots, \alpha^{-1}\gamma_n$ are all the (distinct) conjugates of β over $\mathbb{Q}(\alpha)$ and hence over $\mathbb{Q}$ since $\deg_{\mathbb{Q}}(\beta) = \deg_{\mathbb{Q}(\alpha)}(\beta)$. As β is an algebraic unit, the product $\prod_{i=1}^{n} \alpha^{-1}\gamma_i = \pm 1$. But $\prod_{i=1}^{n} \gamma_i = \pm 1$ because $\alpha\beta$ is an algebraic unit. Hence $\alpha^n = \pm 1$. $\square$

Corollary 1. *Suppose that $deg_{\mathbb{Q}}(\alpha) = deg_{\mathbb{Q}}(\beta) = deg_{\mathbb{Q}(\alpha\beta)}(\alpha) = m$. Then*

$$(\alpha\beta)^m = \pm 1.$$

Proof. The proof is immediate if we set $\tilde{\alpha} = \alpha\beta$ and $\tilde{\beta} = \alpha^{-1}$ and use Lemma 1. $\square$

Remark 2. From Equation 2 and Lemma 1, we deduce that if $\deg_{\mathbb{Q}}(\alpha) = \deg_{\mathbb{Q}}(\beta)$ is coprime to $\deg_{\mathbb{Q}}(\alpha\beta)$, then $(\alpha\beta)^{\deg_{\mathbb{Q}}(\beta)} = \pm 1$.

Corollary 2. *If $deg_{\mathbb{Q}}(\alpha)$ is coprime to $deg_{\mathbb{Q}}(\beta)$ and $deg_{\mathbb{Q}}(\alpha\beta)$ is prime, then*

$$\alpha^{deg_{\mathbb{Q}}(\alpha\beta)} = \pm 1.$$

Proof. If $m = [\mathbb{Q}(\alpha) : \mathbb{Q}]$ is coprime to $n = [\mathbb{Q}(\alpha) : \mathbb{Q}]$, then $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] = n$ and $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\beta)] = m$ (see Equation 2). If $p = [\mathbb{Q}(\alpha\beta) : \mathbb{Q}]$ is prime and if p does not divide m, then p divides n. But

$$n = [\mathbb{Q}(\alpha)(\alpha\beta) : \mathbb{Q}(\alpha)] \leq [\mathbb{Q}(\alpha\beta : \mathbb{Q}] = p,$$

and hence $p = n$. This implies that $\deg_{\mathbb{Q}}(\beta) = \deg_{\mathbb{Q}}(\alpha\beta) = \deg_{\mathbb{Q}(\alpha)}(\beta)$. From Lemma 1, we deduce that $\alpha^p = \pm 1$. $\square$

Corollary 3. *If $deg_{\mathbb{Q}}(\alpha)$ is coprime to $deg_{\mathbb{Q}}(\beta)$ and if $deg_{\mathbb{Q}}(\alpha\beta) = deg_{\mathbb{Q}}(\beta)$, then*

$$\alpha^{deg_{\mathbb{Q}}(\beta)} = \pm 1.$$

Proof. Suppose $\deg_{\mathbb{Q}}(\alpha\beta) = \deg_{\mathbb{Q}}(\beta)$. Then

$$[\mathbb{Q}(\beta) : \mathbb{Q}] = [\mathbb{Q}(\alpha\beta) : \mathbb{Q}] = [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)]$$

since $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ and $[\mathbb{Q}(\beta) : \mathbb{Q}]$ are relatively prime, and $[\mathbb{Q}(\beta) : \mathbb{Q}]$ is greater than or equal to $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)]$; see Remark 1 and Equation (2). Hence

$$\deg_{\mathbb{Q}}(\beta) = \deg_{\mathbb{Q}}(\alpha\beta) = \deg_{\mathbb{Q}(\alpha)}(\beta).$$

From Lemma 1, we deduce that $\alpha^{\deg_{\mathbb{Q}}(\beta)} = \pm 1$. $\square$

Lemma 2. *If $deg_{\mathbb{Q}}(\alpha) < deg_{\mathbb{Q}}(\beta)$ then $deg_{\mathbb{Q}}(\alpha\beta)$ cannot be coprime to $deg_{\mathbb{Q}}(\beta)$.*

Proof. Let $\deg_{\mathbb{Q}}(\alpha) = m, \deg_{\mathbb{Q}}(\beta) = n$ with $m < n$, and let $\deg_{\mathbb{Q}}(\alpha\beta) = l$. Suppose that l is coprime to n. Then by Equation (2), $n = [\mathbb{Q}(\alpha,\beta) : \mathbb{Q}(\alpha\beta)]$ and $l = [\mathbb{Q}(\alpha,\beta) : \mathbb{Q}(\beta)]$ since $[\mathbb{Q}(\beta)(\alpha\beta) : \mathbb{Q}(\beta)] \leq [\mathbb{Q}(\alpha\beta) : \mathbb{Q}]$. But then

$$n = [\mathbb{Q}(\alpha\beta)(\alpha) : \mathbb{Q}(\alpha\beta)] \leq m = [\mathbb{Q}(\alpha) : \mathbb{Q}]$$

which is a contradiction. □

Lemma 3. *If $\deg_{\mathbb{Q}}(\alpha) = 4$, $\deg_{\mathbb{Q}}(\beta) = 6$, and $\deg_{\mathbb{Q}}(\alpha\beta) = 3$, then $(\alpha\beta)^3 = \pm 1$.*

Proof. Let $K = \mathbb{Q}(\alpha,\beta)$. Then we have $[K : \mathbb{Q}(\alpha)] = 3$ and hence $[K : \mathbb{Q}] = 12$. Also $[\mathbb{Q}(\beta) : \mathbb{Q}] = 6$ implies that $[K : \mathbb{Q}(\beta)] = 2$. Let $\alpha_1 = \alpha, \alpha_2$ be the conjugates of α over $\mathbb{Q}(\beta)$ and let $\beta_1 = \beta, \beta_2$ and β_3 denote the conjugates of β over $\mathbb{Q}(\alpha)$. Then the conjugates of $\alpha\beta$ over $\mathbb{Q}(\alpha)$ (and hence over $\mathbb{Q}$) are $\alpha_1\beta_1, \alpha_1\beta_2, \alpha_1\beta_3$. Applying the non-trivial Galois automorphism σ of K over $\mathbb{Q}(\beta)$ to $\alpha_1\beta_1$, we obtain $\alpha_2\beta_1$. Without loss of generality, we assume that

$$\alpha_2\beta_1 = \alpha_1\beta_2 \tag{3}$$

This shows that β_2 and β_3 are in K and hence K over $\mathbb{Q}(\alpha)$ is Galois. We pick a Galois automorphism ρ of K over $\mathbb{Q}(\alpha)$ such that $\rho(\beta_1) = \beta_2, \rho(\beta_2) = \beta_3$ and $\rho(\beta_3) = \beta_1$.

We claim that $\alpha_2 \in \mathbb{Q}(\alpha_1)$. If α_2 is not in $\mathbb{Q}(\alpha_1)$, then $K = \mathbb{Q}(\alpha_1, \alpha_2)$. Applying ρ to α_2, we can conclude that K is a splitting field of the minimal polynomial of α over $\mathbb{Q}$. Its Galois group, a subgroup of order 12 of the symmetric group S_4, must be the alternating group A_4. Note that K contains a splitting field of the minimal polynomial of $\alpha\beta$ over $\mathbb{Q}$, say M. In view of the effect of σ and ρ, the Galois group of M over $\mathbb{Q}$ is S_3. But A_4 does not surject onto S_3 which a contradiction. Hence α_2 is in $\mathbb{Q}(\alpha_1)$.

Since ρ is a Galois automorphism of K over $\mathbb{Q}(\alpha_1)$, $\rho(\alpha_2) = \alpha_2$ and if we apply ρ to Equation 3, we get $\alpha_2\beta_2 = \alpha_1\beta_3$ and $\alpha_2\beta_3 = \alpha_1\beta_1$. This gives $\frac{\beta_1}{\beta_2} = \frac{\beta_3}{\beta_1}$ and hence

$$(\alpha_1\beta_1)^2 = (\alpha_1\beta_2)(\alpha_1\beta_3).$$

Then $(\alpha_1\beta_1)^3 = (\alpha_1\beta_1)(\alpha_1\beta_2)(\alpha_1\beta_3) = \pm 1$. □

Lemma 4. *Let $\deg_{\mathbb{Q}}(\alpha) = 6$ and β denote a conjugate of α over $\mathbb{Q}$, such that $\deg_{\mathbb{Q}(\alpha\beta)}(\alpha) = 3$, then $(\alpha\beta)^3 = \pm 1$.*

Proof. We claim that α and β are not conjugates over $\mathbb{Q}(\alpha\beta)$. Because if α and β are conjugates over $\mathbb{Q}(\alpha\beta)$, then the extension $\mathbb{Q}(\alpha,\beta)$ over $\mathbb{Q}(\alpha\beta)$ is cyclic Galois extension of degree 3 and there exists an automorphism of $\mathbb{Q}(\alpha,\beta)$ of order 3 mapping α to β and fixing $\alpha\beta$, which is impossible. Let $\{\alpha_1 = \alpha, \alpha_2, \alpha_3\}$ and $\{\beta = \beta_1, \beta_2, \beta_3\}$ denote the sets of conjugates of α and β over $\mathbb{Q}(\alpha\beta)$ respectively. Then it can be seen that $\alpha, \beta_2^{-1}\alpha\beta$ and $\beta_3^{-1}\alpha\beta$ are conjugates over $\mathbb{Q}(\alpha\beta)$. Hence $(\alpha_2\beta_2)(\alpha_3\beta_3) = (\alpha\beta)^3$. We conclude that

$$(\alpha\beta)^3 = \prod_{i=1}^{3} \alpha_i \prod_{i=1}^{3} \beta_i = \pm 1$$

since α and β are algebraic units and conjugates over $\mathbb{Q}$. □

4. 13-dimensional Anosov Lie algebras: Proof of Theorem 1

Although Theorem 1 holds for Lie algebras with arbitrary number of steps, for clarity of exposition we write the proof for 2-step nilpotent Lie algebras. The idea of the proof generalizes easily to Lie algebras with three or more steps, although the details are cumbersome.

We note that, to prove the existence of a 13-dimensional indecomposable Anosov Lie algebra, our known methods (from [3] or [8]) are not sufficient. We use the properties of the special algebraic units arising from Anosov automorphisms. If we consider all possible cases, by using Lemmas from Section 3, we rule out all cases for 13-dimensional Lie algebra without an abelian factor to be Anosov except for the type $(9,4)$ (see Definition 2).

We recall [9, Proposition 2.1] and state it for 2-step Anosov Lie algebras which will be used throughout this paper.

Proposition 1. *Let $\mathfrak{n}$ be a 2-step Anosov Lie algebra. Then there exists a hyperbolic semisimple automorphism τ of $\mathfrak{n}$ and a vector space decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus [\mathfrak{n},\mathfrak{n}]$ such that $\tau(\mathfrak{n}_1) = \mathfrak{n}_1$ and the characteristic polynomial f (resp. g) of the restriction of τ to $\mathfrak{n}_1$ (resp. to $[\mathfrak{n},\mathfrak{n}]$) is with integer coefficients.*

Let $\mathfrak{n}$ be a 2-step Anosov Lie algebra and let τ, $\mathfrak{n}_1$, f and g be chosen as in Proposition 1. We note that none of the roots of f and g are of modulus equal to 1. All the roots of g are of the form $\alpha\beta$ where α and β are roots of f. Moreover, f and g are monic polynomials with integer coefficients with constant term ± 1. Hence the roots of f and g are algebraic units. We may assume that the constant term of f and g is 1 by considering square of an automorphism if required.

Lemma 5. *Let $\mathfrak{n}$ be a 2-step Anosov Lie algebra with $\mathfrak{n} = \mathfrak{n}_1 \oplus [\mathfrak{n},\mathfrak{n}]$ and τ be a semisimple hyperbolic automorphism of $\mathfrak{n}$ such that $\tau(\mathfrak{n}_1) = \mathfrak{n}_1$. Let f (resp. g) denote the characteristic polynomial of $\tau|_{\mathfrak{n}_1}$ (resp. $\tau|_{[\mathfrak{n},\mathfrak{n}]}$). If there exists a root α of f such that for all roots β of f one has $g(\alpha\beta) \neq 0$ then $\mathfrak{n}$ has an abelian factor.*

Proof. Since τ is a semisimple automorphism, there exists a basis of $\mathfrak{n}_1^{\mathbb{C}}$ (complexification of $\mathfrak{n}_1$) consisting of the eigenvectors corresponding to the eigenvalues of the restriction of τ on $\mathfrak{n}_1$. Let X be a eigenvector corresponding to the eigenvalue α. Then $[X,Y] = 0$ for all $Y \in \mathfrak{n}_1^{\mathbb{C}}$ by our hypothesis. Hence X belongs to the center of $\mathfrak{n}^{\mathbb{C}}$, $\mathfrak{Z}(\mathfrak{n}^{\mathbb{C}})$. In particular $\mathfrak{Z}(\mathfrak{n}) \cap [\mathfrak{n},\mathfrak{n}] \neq \mathfrak{Z}(\mathfrak{n})$, since $\mathfrak{Z}(\mathfrak{n}^{\mathbb{C}}) \cap [\mathfrak{n}^{\mathbb{C}},\mathfrak{n}^{\mathbb{C}}] = (\mathfrak{Z}(\mathfrak{n}) \cap [\mathfrak{n},\mathfrak{n}])^{\mathbb{C}}$. This implies that $\mathfrak{n}$ has an abelian factor. $\square$

Let $\mathfrak{n}$ be a 2-step 13-dimensional Anosov Lie algebra without an abelian factor type (n_1, n_2). Then $n_1 \geq 5$ since $\mathfrak{n}$ is 13-dimensional and $n_2 \geq 2$ (equivalently $n_1 \leq 11$) because an Anosov automorphism cannot act as 1 or -1 on a one-dimensional center. We consider all the cases and we will show that $\mathfrak{n}$ must be of the type $(9,4)$. We choose τ, $\mathfrak{n}_1$, f and g as in Proposition 1.

Case $(11,2)$. We note that g is irreducible over $\mathbb{Z}$ since the degree of g is 2 and g does not have an eigenvalue of modulus 1. If f is irreducible over $\mathbb{Z}$ then there exist algebraic

units, α and β, such that $\deg_{\mathbb{Q}}(\alpha) = \deg_{\mathbb{Q}}(\beta) = 11$ and $\deg_{\mathbb{Q}}(\alpha\beta) = 2$. This is not possible by Remark 2. If f is reducible over $\mathbb{Z}$, then there exists an odd degree irreducible factor of f over $\mathbb{Z}$, say h. The product of roots of h cannot occur as a root of g by Remark 2. Since $\mathfrak{n}$ is without an abelian factor, we can see that degree of h must be 3 and that there exists an irreducible factor of f over $\mathbb{Z}$, say h', of degree 6; see Corollary 2 and Remark 2. But then $f = hh'h''$ where h'' is an irreducible polynomial of degree 2 over $\mathbb{Z}$. The product $\alpha\beta$, where α is a root of h and β is a root of h'', cannot occur as a root of g by Corollary 2. If α is a root of h' and β is a root of h'', then we have $\deg_{\mathbb{Q}}(\alpha) = 6$ and $\deg_{\mathbb{Q}}(\beta) = 2$. In this case $\deg_{\mathbb{Q}}(\alpha\beta) \neq 2$ as $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha\beta)] \leq 2$ and $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] \geq 6$; see Remark 1 and Equation (2). Thus there exists an abelian factor (see Lemma 5) which is a contradiction. Hence there does not exist an Anosov Lie algebra without an abelian factor of type $(11, 2)$.

Case $(10, 3)$. Also in this case g must be irreducible over $\mathbb{Z}$. Moreover, f cannot be irreducible by Remark 2. If f has two irreducible factors, say f_1 and f_2 over $\mathbb{Z}$ of degree 2 and 8 respectively, and if α is a root of f_2, then by using Remark 2 it can be seen that for all roots β of f_2, $g(\alpha\beta) \neq 0$. Also, by using Lemma 2, we deduce that for all roots β of f_2, $g(\alpha\beta) \neq 0$. This is not possible since $\mathfrak{n}$ has no abelian factor (see Lemma 5). Similarly, considering all possibilities for the factorization of f and using Remark 2, Lemma 2, Corollary 2 and Lemma 5, it can be seen that either $f = f_1 f_2$ where f_1 and f_2 are irreducible factors of f over $\mathbb{Z}$ of degree 4 and 6 respectively; or $f = h_1 h_2 h_3$ where h_1, h_2 and h_3 are irreducible factors of f over $\mathbb{Z}$ such that deg $h_1 =$ deg $h_2 = 2$ and deg $h_3 = 6$.

Suppose that $f = f_1 f_2$ where f_1 and f_2 are irreducible factors of f over $\mathbb{Z}$ of degree 4 and 6 respectively. Since $\mathfrak{n}$ does not have an abelian factor, there exist roots α of f_1 and β of f_2 such that $g(\alpha\beta) = 0$ by Lemma 5 and Remark 2. But this is not possible by Lemma 3.

For the other case, we first prove the following lemma:

Lemma 6. *Let $\beta_1, \beta_3, \beta_3$ denote the distinct roots of an irreducible polynomial of degree 3 over $\mathbb{Z}$ with ± 1 constant term. Let λ and μ be real algebraic units such that*

$$deg_{\mathbb{Q}}(\lambda) = 2 = deg_{\mathbb{Q}}(\mu).$$

If $\{\mu\beta_1, \mu\beta_2, \mu\beta_3, \mu^{-1}\beta_1, \mu^{-1}\beta_2, \mu^{-1}\beta_2\} = \{\lambda\beta_1, \lambda\beta_2, \lambda\beta_3, \lambda^{-1}\beta_1, \lambda^{-1}\beta_2, \lambda^{-1}\beta_2\}$, then either $\mu = \lambda$ or $\mu = \lambda^{-1}$.

Proof. If $\mu \neq \lambda$ and $\mu\beta_1 = \lambda\beta_i$ for some $i \in \{2, 3\}$, then $\beta_1\beta_i^{-1} = \lambda\mu^{-1}$. But $\deg_{\mathbb{Q}}(\lambda\mu^{-1}) \in \{2, 4\}$ and $\deg_{\mathbb{Q}}(\beta_1) = \deg_{\mathbb{Q}}(\beta_i^{-1}) = 3$. By Remark 2, $\lambda^3 = \mu^3$. This is a contradiction because μ and λ are reals and $\mu \neq \lambda$. Similarly we see that if $\mu \neq \lambda^{-1}$ then $\mu\beta_1 \neq \lambda^{-1}\beta_i$ for all $i \in \{2, 3\}$. Hence $\mu = \lambda$ or $\mu = \lambda^{-1}$. □

Now suppose that $f = h_1 h_2 h_3$ where h_1, h_2 and h_3 are irreducible factors of f over $\mathbb{Z}$ such that deg $h_1 =$ deg $h_2 = 2$ and deg $h_3 = 6$. Since h_1 and h_2 are monic polynomials of degree 2 with unit constant term, we may assume that λ and λ^{-1} are roots of h_1, μ and μ^{-1} are roots of h_2. We note that λ and μ are reals since they are of degree 2 algebraic units which are not on the unit circle. Let $\{\alpha_1, \ldots, \alpha_6\}$ denote the set of roots of h_3 and let $\{\beta_1, \beta_2, \beta_3\}$ denote the set of roots of g. Since $\mathfrak{n}$ is without an

abelian factor, there exist α_i and α_j such that $\mu\alpha_i$ and $\lambda\alpha_j$ are roots of g, $1 \leq i, j, \leq 6$. Hence the set of roots of h_3 is $\{\mu\beta_1, \mu\beta_2, \mu\beta_3, \mu^{-1}\beta_1, \mu^{-1}\beta_2, \mu^{-1}\beta_2\}$ which is the same as $\{\lambda\beta_1, \lambda\beta_2, \lambda\beta_3, \lambda^{-1}\beta_1, \lambda^{-1}\beta_2, \lambda^{-1}\beta_2\}$. By Lemma 6, we conclude that either $\mu = \lambda$ or $\mu = \lambda^{-1}$. Without loss of generality we assume that $\mu = \lambda$.

We recall some notations which were introduced at the beginning of this section. $\mathfrak{n} = \mathfrak{n}_1 \oplus [\mathfrak{n}, \mathfrak{n}]$ and τ is the semisimple Anosov automorphism of $\mathfrak{n}$ such that $\tau(\mathfrak{n}_1) = \mathfrak{n}_1$ and f is the characteristic polynomial of $\tau|_{\mathfrak{n}_1}$. Now let $\mathfrak{n}^{\mathbb{C}}$ (resp. $\mathfrak{n}_1^{\mathbb{C}}$) denote the complexification of $\mathfrak{n}$ (resp. $\mathfrak{n}_1$). Let $\{X_1, X_2, Y_1, Y_2, Z_1, Z_2, \cdots, Z_6\}$ denote a basis of $\mathfrak{n}_1^{\mathbb{C}}$ such that $X_1, X_2, Y_1, Y_2, Z_1, Z_2, \cdots, Z_6$ are eigenvectors of $\tau|_{\mathfrak{n}_1}$ corresponding to the eigenvalues $\lambda, \lambda^{-1}, \lambda, \lambda^{-1}, \lambda\beta_1, \lambda\beta_2, \lambda\beta_3, \lambda^{-1}\beta_1, \lambda^{-1}\beta_2, \lambda^{-1}\beta_3$ respectively. Then $X_1 - cY_1$ is in the center of $\mathfrak{n}^{\mathbb{C}}$ for some $c \in \mathbb{C}$, which would generate an abelian factor. Hence there does not exist an Anosov Lie algebra without an abelian factor of type $(10, 3)$.

Case $(8, 5)$**.** By Corollary 2, Corollary 3, Remark 2 and Lemma 2, we can see that $f = f_1 f_2$ and $g = g_1 g_2$ such that $f_1, f_2, g_1\ g_2$ are irreducible polynomials over $\mathbb{Z}$ of degree $2, 6, 2$ and 3 respectively. There exist roots α and β of f such that $g_1(\alpha\beta) = 0$. We note that α and β cannot both be roots of f_1. If $f_1(\alpha) = 0$ and $f_2(\beta) = 0$, then $[\mathbb{Q}(\alpha\beta)(\alpha) : \mathbb{Q}(\alpha\beta)] \in \{1, 2\}$, which gives a contradiction since $[\mathbb{Q}(\alpha\beta) : \mathbb{Q}] = 2$ (see Equation (2)). Hence α and β both should be roots of f_2. We note that $[\mathbb{Q}(\alpha)(\alpha\beta) : \mathbb{Q}(\alpha)] \in \{1, 2\}$. If $[\mathbb{Q}(\alpha)(\alpha\beta) : \mathbb{Q}(\alpha)] = 1$, then $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha\beta)] = 3$. By Lemma 4, we conclude that $(\alpha\beta)^3 = \pm 1$ which is a contradiction. If $[\mathbb{Q}(\alpha)(\alpha\beta) : \mathbb{Q}(\alpha)] = 2$, then $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha\beta)] = 6$ which is a contradiction by Corollary 1.

Case $(7, 6)$. By Corollary 2, Corollary 3, Remark 2 and Lemma 2 we see that either (i) f factors as a product of irreducible polynomials f_1, f_2 and f_3 over $\mathbb{Z}$ such that deg $f_1 =$ deg $f_2 = 2$ and deg $f_3 = 3$ and g is an irreducible polynomial of degree 6 over $\mathbb{Z}$, or (ii) $f = h_1 h_2$ where h_1 is a degree 3 irreducible polynomial and h_2 is a degree 4 irreducible polynomial over $\mathbb{Z}$ and g is an irreducible polynomial over $\mathbb{Z}$.

In (i), let λ and μ denote the roots of f_1 and f_2 respectively. Let $\alpha_1, \alpha_2, \alpha_3$ denote the roots of f_3. Since $\mathfrak{n}$ is without an abelian factor, $\lambda\alpha_i$ and $\mu\alpha_j$ must be roots of g. Hence

$$\{\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3, \lambda\alpha_1^{-1}, \lambda\alpha_2^{-1}, \lambda\alpha_3^{-1}\} = \{\mu\beta_1, \mu\beta_2, \mu\beta_3, \mu\beta_1^{-1}, \mu\beta_2^{-1}, \mu\beta_3^{-1}\}.$$

By Lemma 6, $\mu = \lambda$ or $\mu = \lambda^{-1}$. Suppose $\mu = \lambda$. We recall that f is the characteristic polynomial of $\tau|_{\mathfrak{n}_1}$ where τ is a semisimple Anosov automorphism of $\mathfrak{n}$ such that $\tau(\mathfrak{n}_1) = \mathfrak{n}_1$ and $\mathfrak{n} = \mathfrak{n}_1 \oplus [\mathfrak{n}, \mathfrak{n}]$. Let $\{X_1, X_2, Y_1, Y_2, Z_1, Z_2, Z_3\}$ denote a basis of $\mathfrak{n}_1^{\mathbb{C}}$ (complexification of $\mathfrak{n}_1$) such that $X_1, X_2, Y_1, Y_2, Z_1, Z_2, Z_3$ are the eigenvectors of $\tau|_{\mathfrak{n}_1}$ corresponding to the eigenvalues $\lambda, \lambda^{-1}, \mu, \mu^{-1}, \alpha_1, \alpha_2, \alpha_3$ respectively. Then $X_1 - aY_1$ is in the center of $\mathfrak{n}^{\mathbb{C}}$ which is a contradiction to our assumption that $\mathfrak{n}$ has no abelian factor.

In (ii), if α and β are roots of f such that $g(\alpha\beta) = 0$, then α and β cannot be both roots of h_1 since $\deg_{\mathbb{Q}}(\alpha\beta) = 6$. We can assume that $h_1(\alpha) = 0 = h_2(\beta)$. Then by applying Lemma 3 to $\tilde{\alpha} = \beta^{-1}$ and $\tilde{\beta} = \alpha\beta$, we see that $\alpha^3 = \pm 1$, which is a contradiction. Hence we conclude that there is no Anosov Lie algebra of type $(7, 6)$ without an abelian factor.

Case $(6,7)$**.** By Remark 2, Lemma 2 and Lemma 5, we can see that the only possibility in this case is the following: f is irreducible over $\mathbb{Z}$, and $g = g_1 g_2$ where g_1 and g_2 are irreducible over $\mathbb{Z}$ of degree 3 and 4 respectively. In this case we will prove that there does not exist a root $\alpha\beta$ of g_2 such that α and β are roots of f. Suppose that α and β are roots of f such that $\alpha\beta$ is a root of g_2. Let $n = [\mathbb{Q}(\alpha\beta)(\alpha) : \mathbb{Q}(\alpha\beta)]$. Then we note that $n \in \{3, 6\}$ (by Remark 1 and Equation (2)). If $n = 6$, we get a contradiction by Lemma 1. Hence $n = 3$. But then by Lemma 4, $(\alpha\beta)^3 = 1$, which is a contradiction.

Case $(5,8)$**.** By Remark 2 and Lemma 2 it can be seen that there does not exist an Anosov Lie algebra without an abelian factor and of type $(5,8)$.

Case $(9,4)$**.** By Remark 2 and Lemma 2, we see that there are two possibilities in this case: (i) $f = f_1 f_2$ where f_1 and f_2 are irreducible polynomials over $\mathbb{Z}$ of degree 3 and 6 respectively, and g is irreducible over $\mathbb{Z}$. (ii) $f = h_1 h_1$ and $g = g_1 g_2$ such that h_1, h_2, g_1 and g_2 are irreducible polynomials over $\mathbb{Z}$ and deg $h_1 = 3$, deg $h_2 = 6$ and deg $g_1 =$ deg $g_2 = 2$.

In case (i), there exists roots α of f_1 and β of f_2 such that $g(\alpha\beta) = 0$ (by Remark 2 and Lemma 5). By applying Lemma 3 to $\tilde{\alpha} = \alpha\beta$ and $\tilde{\beta} = \beta^{-1}$, we conclude that $\alpha^3 = \pm 1$, a contradiction. Hence case (i) is not possible.

In case (ii), we will prove that there exists an indecomposable Anosov Lie algebra i.e. an Anosov Lie algebra which cannot be written as a direct product of two proper ideals (see [3]).

Suppose that $f = h_1 h_1$ and $g = g_1 g_2$ such that h_1, h_2, g_1 and g_2 are irreducible polynomials over $\mathbb{Z}$ and deg $h_1 = 3$, deg $h_2 = 6$ and deg $g_1 =$ deg $g_2 = 2$. Let $\{\alpha_1, \alpha_2, \alpha_3\}$, $\{\lambda, \lambda^{-1}\}$ and $\{\mu, \mu^{-1}\}$ denote the sets of roots of h_1, g_1 and g_2 respectively. Since there is no abelian factor, there exist α and β, roots of h_1 and h_2 respectively, such that $\alpha\beta = \lambda$. Similarly there exist α' and β', roots of h_1 and h_2 respectively, such that $\alpha'\beta' = \mu$. Then it can be seen that the set of roots of h_2 is given by

$$\{\lambda\alpha_1^{-1}, \lambda\alpha_2^{-1}, \lambda\alpha_3^{-1}, \lambda^{-1}\alpha_1^{-1}, \lambda^{-1}\alpha_2^{-1}, \lambda^{-1}\alpha_3^{-1}\} =$$
$$\{\mu\alpha_1^{-1}, \mu\alpha_2^{-1}, \mu\alpha_3^{-1}, \mu^{-1}\alpha_1^{-1}, \mu^{-1}\alpha_2^{-1}, \mu^{-1}\alpha_3^{-1}\}.$$

Hence by Lemma 6, $\mu = \lambda$ or $\mu = \lambda^{-1}$. Without loss of generality we assume that $\mu = \lambda$.

Let $\mathfrak{n}^{\mathbb{C}}$ $(\mathfrak{n}_1^{\mathbb{C}})$ denote the complexification of $\mathfrak{n}$ ($\mathfrak{n}_1$ respectively). Let X_1, X_2, and X_3 denote the eigenvectors (in $\mathfrak{n}_1^{\mathbb{C}}$) of $\tau|_{\mathfrak{n}_1}$ corresponding to the eigenvalues α_1, α_2, and α_3 respectively. Let $Y_1, Y_2, \ldots, Y_6$ denote the eigenvectors in $\mathfrak{n}_1^{\mathbb{C}}$ corresponding to the eigenvalues $\lambda\alpha_1^{-1}, \lambda\alpha_2^{-1}$, $\lambda\alpha_3^{-1}, \lambda^{-1}\alpha_1^{-1}$, $\lambda^{-1}\alpha_2^{-1}, \lambda^{-1}\alpha_3^{-1}$ respectively. Let Z_1, W_1 be linearly independent eigenvectors corresponding to an eigenvalue λ and let Z_2, W_2 be linearly independent eigenvectors corresponding to an eigenvalue λ^{-1}. We may assume that the Lie brackets are given by the following relations:

$$\begin{aligned} &[X_1, Y_1] = Z_1, && [X_2, Y_2] = W_1 \\ &[X_1, Y_4] = Z_2, && [X_2, Y_5] = W_2 \\ &[X_3, Y_3] = aZ_1 + bW_1, && [X_3, Y_6] = cZ_2 + dW_2 \end{aligned} \tag{4}$$

where $a, b, c, d \in \mathbb{C}$ and other commutators in the generators vanish. This completes the proof of the main Theorem 1 for 2-step Anosov Lie algebras. As noted in the beginning of this section, the same idea leads to a proof of the general theorem for Lie algebras with arbitrary number of steps. In the general proof, Proposition 1 has to be replaced by [9, Proposition 2.1]

Remark 3. For an actual example of a 13-dimensional Anosov Lie algebra, see [13].

5. More nonexistence results

As noted in the introduction (§1), it is known that there does not exist a non-toral 7-dimensional Anosov Lie algebra (see [10]). Here we give an alternative proof of this fact using the results from §3. First we note that a 7-dimensional Anosov Lie algebra has to be 2-step and of the type $(4, 3)$ or $(5, 2)$ (see [9, Proposition 2.3]).

As in the 13-dimensional case (§4), following Proposition 1, we choose a decomposition of a 7-dimensional Anosov Lie algebra $\mathfrak{n}$ as $\mathfrak{n} = \mathfrak{n}_1 \oplus [\mathfrak{n}, \mathfrak{n}]$ and a hyperbolic semisimple automorphism τ such that $\tau(\mathfrak{n}_1) = \mathfrak{n}_1$ such that the characteristic polynomial f (resp. g) of the restriction of τ to $\mathfrak{n}_1$ (resp. to $[\mathfrak{n}, \mathfrak{n}]$) is with integer coefficients. Then we note that the roots of g are certain products of the roots of f. The roots of f and g are algebraic units and none of them is of absolute value 1.

Case $(4, 3)$. In this case, g is of degree 3 and hence has to be irreducible. Now either f is irreducible or f is a product of two degree 2 irreducible polynomials. By using Remark 2, we can see that both cases are impossible.

Case $(5, 2)$. In this case also g is irreducible and is of degree 2. It can be seen (using Remark 2) that f is reducible and f is a product of two irreducible polynomials f_1 and f_2 such that f_1 is of degree 2 and f_2 is of degree 3. Now if α is a root of f_1 and β is a root of f_2 then $\alpha\beta$ will not occur as a root of g by Lemma 2. Moreover, the product of two roots of f_1 is ± 1 and the product of two roots of f_2 cannot be of degree 2 by Remark 2. Hence this case is also impossible.

In a similar way, we can use Corollary 3 and Remark 2 to give a simpler proof to show that there are no Anosov Lie algebras of type $(5, 3)$ with no abelian factor and there are no Anosov Lie algebras of type $(3, 3, 2)$ with no abelian factor; this was first proved in [9].

6. Anosov Lie algebras of type $(n, 2)$, n odd

In this section we study 2-step Anosov Lie algebra of type $(n, 2)$ (See Definition 2), where n is odd. Let $\mathfrak{n}$ be a 2-step Anosov Lie algebra of type $(n, 2)$. Consider a vector space decomposition $\mathfrak{n} = \mathfrak{n}_1 \oplus [\mathfrak{n}, \mathfrak{n}]$ of $\mathfrak{n}$. We say that $\mathfrak{n}_1$ is *decomposable* if $\mathfrak{n}_1 = V_1 \oplus V_2$ where V_1 and V_2 are nontrivial subspaces of $\mathfrak{n}_1$ such that $[V_1, V_2] = \{0\}$. We will prove that if a 2-step nilpotent Lie algebra $\mathfrak{n}$ of type $(n, 2)$, with n odd, is Anosov then there exists $\mathfrak{n}_1$ (such that $\mathfrak{n} = \mathfrak{n}_1 \oplus [\mathfrak{n}, \mathfrak{n}]$) which is decomposable.

From now onwards we will assume that $\mathfrak{n}$ is a 2-step nilpotent Lie algebra of type $(n, 2)$ such that n is odd. Suppose that $\mathfrak{n}$ is an Anosov Lie algebra. Now since $\mathfrak{n}$ is Anosov 2-step, we choose $\tau, \mathfrak{n}_1, f$ and g as in Proposition 1. We note that the roots of g are real since deg $g = 2$ and τ is hyperbolic.

Let α denote a real root of f such that $\deg_{\mathbb{Q}}(\alpha)$ is odd. Since n is odd, α exists. Let μ denote a root of g. Since $\deg_{\mathbb{Q}}(\mu) = 2$ and μ is an algebraic unit, μ^{-1} is also a root of g. Let S denote the set of eigenvalues of $\tau|_{\mathfrak{n}_1}$ which are contained in $\{\mu^{2n}\alpha, \mu^{2n+1}\alpha^{-1} : n \in \mathbb{Z}\}$. Let V_1 denote the sum of the eigenspaces corresponding to all eigenvalues in S and let V_2 denote the sum of the eigenspaces corresponding the eigenvalues of $\tau|_{\mathfrak{n}_1}$ which are not contained in S. Since $\tau|_{\mathfrak{n}_1}$ is semisimple, $\mathfrak{n}_1 = V_1 \oplus V_2$. Since α is an eigenvalue of $\tau|_{\mathfrak{n}_1}$, V_1 is nontrivial. Let β be a conjugate of α over $\mathbb{Q}$, $\beta \neq \alpha$. Then $\deg_{\mathbb{Q}}(\beta) = \deg_{\mathbb{Q}}(\alpha)$ which is odd. We note that β is an eigenvalue of $\tau|_{\mathfrak{n}_1}$ which is not contained in S because $\deg_{\mathbb{Q}}((\mu)^l) = 2$ for all $l \neq 0$ (see Corollary 2). Hence V_2 is nontrivial. Since the eigenvalues of $\tau|_{\mathfrak{n}_2}$ are of the form μ and μ^{-1}, $[V_1, V_2] = \{0\}$. This proves that $\mathfrak{n}_1$ is decomposable. Hence we have proved the following:

Proposition 2. *If $\mathfrak{n}$ is a 2-step Anosov nilpotent Lie algebra of an odd dimension such that dim $[\mathfrak{n}, \mathfrak{n}] = 2$, then there exists a decomposable $\mathfrak{n}_1$, a vector space complement of $[\mathfrak{n}, \mathfrak{n}]$ in $\mathfrak{n}$.*

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